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Magnetic Attitude Control System for Dual-Spin Satellites

KYLE T. ALFRIEND*
Naval Research Laboratory, Washington, D.C.

A closed-loop control law is developed for a dual-spin satellite control system which utilizes the interaction of the geomagnetic field with the satellite dipole parallel to the spin axis. The control law consists of the linear combination of the pitch axis component of the rate of change of the geomagnetic field and the product of the roll angle and roll axis component of the geomagnetic field. Application of the method of multiple time scales yields approximate solutions for the feedback gains in terms of the system parameters. Approximate solutions are also obtained for the response of the system to disturbance torques. A comparison of the approximate solutions and numerical solutions obtained by numerical integration of the exact equations of motion is then given.

Introduction

INTERACTION between onboard magnets and the geomagnetic field has been used as a means of satellite attitude control.[1–7] It has been used extensively as a method for despinning satellites, fine attitude control,[1–5] and recently has been suggested as a means for acquisition for tumbling satellites.[6–7] The development of a suitable control law for the satellite dipole has been the main concern of the previous investigations. Renard[2] was concerned with obtaining the control law for attitude control which would be activated by ground command. The results of his study showed that the quarter-orbit bang-bang control was the best for near magnetic polar orbits. Wheeler[4] applied the method of averaging to obtain the control law for a closed-loop control system. Shigera[5] developed an on-off control law for a single spin satellite which required switching four times per orbit but not every quarter orbit as did the control system developed by Renard.[2]

This investigation is concerned with the development of the control law governing the strength of a satellite dipole parallel to the spin axis. The interaction of this dipole with the geomagnetic field will provide the attitude control of a dual spin satellite. The control system is to be closed-loop as compared to the open-loop system of Renard.[2] The available data from sensors for development of the control law are the roll angle from the horizon sensor and the geomagnetic field components obtained from the magnetometer. As will be shown later $\dot{B}_\theta$, the rate of change of the geomagnetic field along the pitch axis, is proportional to the yaw and roll rates for small angles. The control law, which was proposed by R. Z. Fowler, President of Ithaco, Inc., is

$$M_\theta = K_1 B_\phi \phi - K_2 \dot{B}_\theta \tag{1}$$

where B_ϕ is the component of the geomagnetic field along the roll axis, ϕ is the roll angle, M_θ is the strength of the pitch magnet, and K_1 and K_2 are the feedback gains. The equations of motion for small angles are linear but the coefficients are time

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* Formerly at Ithaco, Inc., Ithaca, N.Y. Presently Consultant, Space Systems Division. Member AIAA.

varying so the classical methods for the design of feedback control systems cannot be used. In this investigation the method of multiple time scales[8-10] is used to obtain the feedback gains K_1 and K_2 which will give the desired system behavior. The response of the system to disturbance torques is then investigated. Finally, the approximate solutions developed from the method of multiple time scales are compared with results obtained from numerical integration of the equations of motion.

Collins and Bonello[7] investigated this problem with a different control law. They obtained an estimate of the yaw rate by putting the roll signal through an appropriate network; however, they obtained no analytical results for the feedback gains.

Equations of Motion

The equations of motion of a dual spin satellite which has a rotor with angular momentum h parallel to the spin axis are developed in the Appendix. For the development of the control law, it is assumed that only small deviations occur in yaw and roll about the equilibrium configuration of the angular momentum perpendicular to the orbit plane and the slowly spinning portion rotating at orbital rate so that the satellite is earth pointing. It is also assumed that the geomagnetic field can be represented by a tilted dipole. The linearized equations of motion are

$$\begin{aligned} I_\psi \ddot{\psi} + K_2 B_\phi{}^2 \dot{\psi} + (h - I_\phi \omega_o + 0.5 K_2 B_\phi B_\psi)\omega_o \psi + \\ [h - (I_\phi + I_\psi)\omega_o - K_2 B_\phi B_\psi]\dot{\phi} + \\ (K_1 + 2\omega_o K_2) B_\phi{}^2 \phi = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} I_\phi \ddot{\phi} + K_2 B_\psi{}^2 \dot{\phi} + [h\omega_o - I_\psi \omega_o{}^2 - (K_1 + 2\omega_o K_2) B_\phi B_\psi]\phi - \\ [h - (I_\phi + I_\psi)\omega_o + K_2 B_\phi B_\psi]\dot{\psi} - 0.5 K_2 \omega_o B_\psi{}^2 \psi = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} B_\phi &= -2\bar{B}_o \sin(\omega_o t + \alpha) \\ B_\theta &= \bar{B}_o \cos(\omega_o t + \alpha) \end{aligned} \tag{4}$$

ψ and ϕ are the yaw and roll angles, I_ψ and I_ϕ are the corresponding moments of inertia, h is the total momentum bias, ω_o is the orbital rate, B_ψ and B_ϕ are the yaw and roll components of the geomagnetic field, and $\bar{B}_o$ and α are constants defined in the Appendix.

With no control ($K_1 = K_2 = 0$), the characteristic equation of the system for small ω_o is

$$\lambda^4 + (\omega_o{}^2 + h^2/I_\phi I_\psi)\lambda^2 + h^2\omega_o{}^2/I_\phi I_\psi = 0 \tag{5}$$

The natural frequencies of this 2 degree-of-freedom system are the nutation frequency $h/(I_\phi I_\psi)^{1/2}$ and the orbital frequency ω_o. If there is no nutation an error in roll becomes an error in yaw of quarter of an orbit later. This mode will be called the orbital mode.

The problem now is to determine the feedback gains K_1 and K_2 so that the desired behavior of the system is obtained. The $K_2 \dot{B}_\theta$ portion of the control law controls the nutation mode of the satellite; i.e., it damps out the nutation so that the angular momentum vector and spin axis coincide. The $K_1 B_\phi \phi$ term controls the orbital mode in that it provides the necessary torque to rotate the spin axis (and angular momentum vector) into alignment with the orbit normal. Since the system has time varying (periodic) coefficients, the classical methods of feedback control system analysis are not applicable. Floquet theory[11] can be used to determine the stability but this is a numerical procedure and does not yield K_1 and K_2 as functions of the system parameters. There are 2 time scales in the system; the one associated with the nutation mode and the one associated with the orbital mode. For practical values of h, I_ϕ, and I_ψ, the orbital frequency is considerably smaller than the nutation frequency, hence the method of multiple time scales[8-10] can be used to obtain an approximate solution to the problem.

Application of the Method of Multiple Time Scales

Before introducing the 2 time scales, it is advantageous to rewrite the equations of motion in dimensionless form. Let the dimensionless time $\bar{t}$ be the nutation mode time scale

$$\bar{t} = \frac{h}{(I_\phi I_\psi)^{1/2}}(t + \alpha/\omega_o) \tag{6}$$

and

$$\varepsilon = (I_\phi I_\psi)^{1/2}\omega_o/h \tag{7}$$

$$a = (I_\psi/I_\phi)^{1/2}, \quad b = (I_\phi/I_\psi)^{1/2} \tag{8}$$

$$\bar{K}_1 = K_1 + 2\omega_o K_2 \tag{9}$$

$$\frac{K_2 \bar{B}_o{}^2}{h} = \varepsilon K_2{}^*, \quad \frac{\bar{K}_1 \bar{B}_o{}^2}{h^2}(I_\phi I_\psi)^{1/2} = \varepsilon K_1{}^* \tag{10}$$

The equations of motion become

$$\begin{aligned} \psi'' + 0.5\varepsilon b K_2{}^*(1 + \cos 2\varepsilon\bar{t})\psi' + \varepsilon b[1 - \varepsilon b + \varepsilon K_2{}^* \sin 2\varepsilon\bar{t}]\psi + \\ [b - \varepsilon(1 + b^2) + \varepsilon K_2{}^* b \sin 2\varepsilon t]\phi' + \\ 0.5\varepsilon b K_1{}^*(1 + \cos 2\varepsilon\bar{t})\phi = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} \phi'' + 2\varepsilon a K_2{}^*(1 - \cos 2\varepsilon\bar{t})\phi' + \varepsilon a(1 - \varepsilon a + K_1{}^* \sin 2\varepsilon\bar{t})\phi - \\ [a - \varepsilon(1 + a^2) - \varepsilon a K_2{}^* \sin 2\varepsilon\bar{t}]\psi' - \\ \varepsilon^2 a K_2{}^*(1 - \cos 2\varepsilon\bar{t})\psi = 0 \end{aligned} \tag{12}$$

where primes denote differentiation with respect to $\bar{t}$. The parameter ε, which is the ratio of the orbital frequency and nutation frequency, is small. Equations (9) and (10) state that the control torque has a small effect on the nutation mode; i.e., the coefficient of critical damping in the nutation mode is small. An approximate series solution in powers of ε will now be obtained using the method of multiple time scales. The independent variable $\bar{t}$ is now replaced by 2 independent time scales ξ and η

$$\xi = \bar{t}; \quad \eta = \varepsilon\bar{t} \tag{13}$$

ξ is the time scale of the nutation mode and η represents the "slow" time scale of the orbital mode. The derivatives with respect to $\bar{t}$ become

$$\begin{aligned} \frac{d}{d\bar{t}} &= \frac{\partial}{\partial\xi} + \varepsilon\frac{\partial}{\partial\eta} \\ \frac{d^2}{d\bar{t}^2} &= \frac{\partial^2}{\partial\xi^2} + 2\varepsilon\frac{\partial^2}{\partial\xi\,\partial\eta} + \varepsilon^2\frac{\partial^2}{\partial\eta^2} \end{aligned} \tag{14}$$

Set

$$\begin{aligned} \psi &= \sum_{n=0}^{\infty} \varepsilon^n \psi_n = \psi_o + \varepsilon\psi_1 + \cdots \\ \phi &= \sum_{n=0}^{\infty} \varepsilon^n \phi_n = \phi_o + \varepsilon\phi_1 + \cdots \end{aligned} \tag{15}$$

Substituting Eqs. (14) and (15) into Eqs. (11) and (12) and equating like powers of ε gives

Zeroth-order ε^0

$$\psi_{0_{\xi\xi}} + b\phi_{0_\xi} = 0; \quad \phi_{0_{\xi\xi}} - a\psi_{0_\xi} = 0 \tag{16}$$

First-order ε^1

$$\begin{aligned} \psi_{1_{\xi\xi}} + b\phi_{1_\xi} = -2\psi_{0_{\xi\eta}} - b\phi_{0_\eta} - \frac{K_2{}^* b}{2}(1 + \cos 2\eta)\psi_{0_\xi} - \\ b\psi_0 + (1 + b^2)\phi_{0_\xi} - K_2{}^* b \sin 2\eta\,\phi_{0_\xi} - \\ \frac{K_1{}^* b}{2}(1 + \cos 2\eta)\phi_0 \end{aligned} \tag{17a}$$

$$\begin{aligned} \phi_{1_{\xi\xi}} - a\psi_{1_\xi} = -2\phi_{0_{\xi\eta}} + a\psi_{0_\eta} - 2aK_2{}^*(1 - \cos 2\eta)\phi_{0_\xi} - \\ (1 + K_2{}^* \sin 2\eta)a\phi_0 - (1 + a^2)\psi_{0_\xi} - aK_2{}^* \sin 2\eta\,\psi_{0_\xi} \end{aligned} \tag{17b}$$

where the subscripts ξ and η denote partial differentiation.

The solutions to Eq. (16) are

$$\begin{aligned} \psi_0 &= A_0(\eta)\cos[\xi + \beta_0(\eta)] + C_0(\eta) \\ \phi_0 &= aA_0(\eta)\sin[\xi + \beta_0(\eta)] + D_0(\eta) \end{aligned} \tag{18}$$

where A_0, β_0, C_0, and D_0 are functions of the time η. For a uniformly valid solution $(\psi_1{}^2 + \phi_1{}^2)/(\psi_0{}^2 + \phi_0{}^2)$ must remain bounded for all time and for asymptotic stability $\lim_{\bar{t}\to\infty} \phi_0 = 0$ and $\lim_{\bar{t}\to\infty} \psi_0 = 0$.

Substitution of the zeroth-order solution Eq. (18) into the first-order Eqs. (17) gives

$$\psi_{1_{\xi\xi}}+b\phi_{1_\xi}=[\beta_{0_\eta}+a-K_2{}^*\sin 2\eta]A_0\cos\xi+[A_{0_\eta}+(-K_1{}^*+bK_2{}^*)(1+\cos 2\eta)A_0/2]\sin\xi-b[D_{0_\eta}+C_0+K_1{}^*(1+\cos 2\eta)D_0/2] \quad (19a)$$

$$\phi_{1_{\xi\xi}}-a\psi_{1_\xi}=[-aA_{0_\eta}-2a^2K_2{}^*(1-\cos 2\eta)A_0]\cos\xi+[a\beta_{0_\eta}+aK_2{}^*\sin 2\eta+(1-a^2K_1{}^*\sin 2\eta)]\sin\xi+a[C_{0_\eta}-D_0(1+K_1{}^*\sin 2\eta)] \quad (19b)$$

The $\cos\xi$ and $\sin\xi$ as well as the constant terms produce resonance or secular terms in ψ_1 and ϕ_1 since the frequencies are equal to the natural frequencies, zero and unity, of the system represented by Eq. (18). For a uniformly valid solution, these secular terms must be eliminated. Their elimination requires

$$\beta_{0_\eta}=[aK_1{}^*\sin 2\eta-a-b]/2 \quad (20)$$

$$A_{0_\eta}+(L_1+L_2\cos 2\eta)A_0=0 \quad (21)$$

$$L_1=(a+b/4)K_2{}^*-K_1{}^*/4 \quad (22a)$$

$$L_2=-(a-b/4)K_2{}^*-K_1{}^*/4 \quad (22b)$$

$$D_{0_\eta}+C_0+K_1{}^*(1+\cos 2\eta)D_0/2=0 \quad (23a)$$

$$C_{0_\eta}-D_0(1+K_1{}^*\sin 2\eta)=0 \quad (23b)$$

The solution to the β_0 equation is not needed since it only gives the small change in frequency resulting from the higher order terms. The solution to Eq. (21) is

$$A_0=\bar{A}_0\exp(-L_1\eta)\exp(-0.5L_2\sin 2\eta) \quad (24)$$

where $\bar{A}_0$ is the constant of integration. For stability $L_1>0$ and the time constant of the nutation mode is $\eta_n=1/L_1$ or $t_n=1/\omega_0 L_1$. In terms of K_1 and K_2 the stability requirement and time constant are

$$K_1<[(4I_\psi+I_\phi)h/(I_\phi I_\psi)-2\omega_0]K_2 \quad (25)$$

$$t_n=\left(\frac{4h}{\bar{B}_0{}^2}\right)\Bigg/\left\{\left[\frac{(4I_\psi+I_\phi)h}{I_\phi I_\psi}-2\omega_0\right]K_2-K_1\right\}\text{ sec} \quad (26a)$$

or

$$t_n=\left(\frac{2\omega_0 h}{\pi\bar{B}_0{}^2}\right)\Bigg/\left\{\left[\frac{(4I_\psi+I_\phi)h}{I_\phi I_\psi}-2\omega_0\right]K_2-K_1\right\}\text{ orbits} \quad (26b)$$

Although the requirement $L_1>0$ guarantees that Eq. (21) is asymptotically stable, if $|L_2|>L_1$ the nutation angle increases for those values of η for which $L_1+L_2\cos 2\eta<0$. To insure that the nutation angle always decreases one should use the requirement $L_1+L_2>0$ which yields

$$K_1<[(h/I_\psi)-2\omega_0]K_2 \quad (27)$$

Now consider the stability of the orbital mode which is governed by Eq. (23)

$$C_{0_\eta}=D_0(1+K_1{}^*\sin 2\eta)$$
$$D_{0_\eta}=-C_0-K_1{}^*(1+\cos 2\eta)D_0/2 \quad (28)$$

Choose as a Lyapunov function

$$V=D_0{}^2+[C_0+K_1{}^*(1-\cos 2\eta)D_0/2]^2 \quad (29)$$

V is positive definite and bounded above by $W_1=3C_0{}^2+4D_0{}^2$ and bounded below by $W_2=(3/4)(C_0{}^2+D_0{}^2)$. Differentiation of Eq. (28) gives

$$\frac{dV}{d\eta}=-\frac{K_1{}^*}{2}[C_0+K_1{}^*(1-\cos 2\eta)D_0/2]^2(1-\cos 2\eta) \quad (30)$$

which is semipositive definite and is equal to zero at the isolated points in time $\eta=(2k+1)\pi/2$, $k=0,1,2\ldots$. Thus by the usual theorems on Liapunov stability the system is asymptotically stable.

To determine the time constant for this mode the method of multiple time scales will be used to obtain an approximate solution for Eq. (28) assuming $K_1{}^*$ is small. Set

$$\alpha=\eta;\qquad \mu=K_1{}^*\eta \quad (31)$$

$$C_0(\eta)=C_{00}(\alpha,\mu)+K_1{}^*C_{01}(\alpha,\mu)+\cdots$$
$$D_0(\eta)=D_{00}(\alpha,\mu)+K_1{}^*D_{01}(\alpha,\mu)+\cdots \quad (32)$$

Substituting Eq. (31) and (32) into Eq. (28) and equating like powers of $K_1{}^*$ gives

0th order ($K_1{}^{*o}$)

$$C_{00_\alpha}=D_{00} \quad (33)$$
$$D_{00_\alpha}=-C_{00}$$

1st order ($K_1{}^{*1}$)

$$C_{01_\alpha}=D_{01}-C_{00_\mu}-D_{00}\sin 2\alpha$$
$$D_{01_\alpha}=-C_{01}-D_{00_\mu}-\frac{D_{00}}{2}(1+\cos 2\alpha) \quad (34)$$

where the subscripts α and μ denote partial differentiation.

The solution to the zeroth-order equations is

$$C_{00}=E_o(\mu)\cos\alpha+F_0(\mu)\sin\alpha$$
$$D_{00}=-E_o(\mu)\sin\alpha+F_0(\mu)\cos\alpha \quad (35)$$

Substituting Eq. (35) into (34) gives

$$C_{01_\alpha}=D_{01}+\left(-E_{0_\mu}-\frac{E_0}{2}\right)\cos\alpha+\left(-F_{0_\mu}+\frac{F_0}{2}\right)\sin\alpha+\frac{E_0}{2}\cos 3\alpha+\frac{F_0}{2}\sin 3\alpha$$

$$D_{01_\alpha}=-C_{01}+(-F_{0_\mu}-(3/4)F_0)\cos\alpha+\left(E_{0_\mu}+\frac{E_0}{4}\right)\sin\alpha-\frac{E_0}{4}\sin 3\alpha-\frac{F_0}{4}\cos 3\alpha \quad (36)$$

The $\cos\alpha$ and $\sin\alpha$ terms result in resonance or secular terms in C_{01} and D_{01}. Setting the coefficients of these secular terms to zero gives

$$E_{0_\mu}+(3/8)E_0=0$$
$$F_{0_\mu}+(1/8)F_0=0 \quad (37)$$

which has the solution

$$E_0=\bar{E}_0\exp[-(3/8)\mu]=\bar{E}_0\exp[-(3/8)K_1{}^*\eta]$$
$$F_0=\bar{F}_0\exp[-(1/8)\mu]=\bar{F}_0\exp[-(1/8)K_1{}^*\eta] \quad (38)$$

Thus, associated with the orbital mode there are two time constants which are approximately a ratio of 3. Obviously the longer time constant will dominate and should be used as the design criteria. The time constants are

$$t_{01}=\left(\frac{4h\omega_0}{\pi\bar{B}_0{}^2}\right)\Bigg/(K_1+2\omega_0K_2)\text{ orbits} \quad (39)$$

$$t_{02}=t_{01}/3 \quad (40)$$

Obviously the accuracy of these approximate time constants is dependent on the value of $K_1{}^*$ since the solution is a series expansion in powers of $K_1{}^*$. However, further analysis shows that in the expressions given for t_{o1} and t_{o2} terms of $O(K_1{}^{*2})$ do not appear; hence t_{o1} and t_{o2} are accurate to $O(K_1{}^{*3})$.

If one considers only motion in the orbital mode then $\bar{E}_0=\psi_i$, $\bar{F}_0=\phi_i$, where ψ_i and ϕ_i are the angles when the satellite is over the equator. Thus t_{01} is the time constant for the mode when ϕ is a maximum at the equator, and t_{02} is the time constant for the mode when ψ has its maximum value at the equator. Therefore, the condition $\phi=\phi_i$, $\psi=0$ at the equator is the most critical condition.

Comparison with Numerical Results

Figure 1 gives a comparison between the approximate solution and the exact solution obtained by numerical integration of the exact equations of motion. Figure 1a gives the envelope of the motion which involves a high frequency motion superimposed on an exponential decay with a varying time constant. The parameters chosen were those of the ITOS satellite in a circular orbit of altitude 1462 km and an inclination of 101.7°. The products of inertia and the effect of the rotation of the earth are included. The values of the parameters are $I_\psi=155.3$ kg-m^2, $I_\phi=135.5$ kg-m^2, $I_\theta=138.9$ kg-m^2, $I_{\psi\phi}=1.1$ kg-m^2, $I_{\psi\theta}=6.4$ kg-m^2, $I_{\phi\theta}=1.6$ kg-m^2, $h=26.6$ kg-m^2/sec, $K_1=5\times10^6$ pole-cm, $K_2=5\times10^7$ pole-cm/gauss/sec. Since the frequencies of the two modes are widely separated, a com-

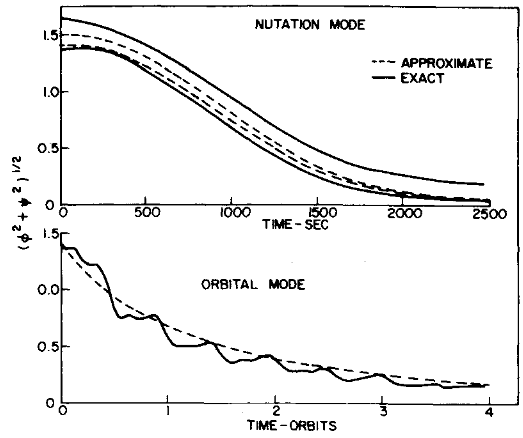

Fig. 1 Comparison of approximate and exact solution.

parison has been made for each mode. This was done by choosing the appropriate initial conditions. Figure 1 shows that the approximate solution adequately describes the motion particularly the decay portion. The difference in the exact and approximate solutions are due to using only the zeroth-order solution, neglecting the products of inertia and rotation of the earth, and linearization of the equations of motion.

Comparison with Floquet Theory

Since Eqs. (2) and (3) or Eqs. (11) and (12), which govern the motion of the system, are linear differential equations with periodic coefficients, their stability can be determined using Floquet theory. To apply Floquet theory to this system, Eqs. (2) and (3) were rewritten in the form $\dot{\mathbf{y}} = A(t)\mathbf{y}$ and the differential equation $\dot{\Phi} = A\Phi$, $\Phi(0,0) = I$ was integrated over the interval $(0, \pi\omega_o)$. The characteristic exponents ρ_i were obtained via

$$\rho_i = \frac{\omega_o}{\pi} \ln \sigma_i \tag{41}$$

where the σ_i are the eigenvalues of $\Phi(\pi/\omega_o, 0)$. The time constants of the system then were obtained by

$$t_c = \frac{-1}{R(\rho_i)} = \frac{\pi}{\ln|\sigma_i|} \text{seconds}$$

or

$$t_c = -1/(\pi \ln|\sigma_i|) \text{ orbits} \tag{42}$$

The comparison is made for the ITOS satellite except a value of $h = 4$ kg-m^2/s is used rather than the ITOS value in order to reduce the computer costs since the number of integration steps increases linearly with h. For the numerical values used $\bar{B} = 0.167$ gauss; $\omega_0 = 9.1 \times 10^{-4}$ sec^{-1}; and $\varepsilon = 0.033$. For stability $K_2 > 7.04\,K_1$.

The comparison given in Table 1 shows that the approximate equations for the time constants give good results except when K_2 is close to the unstable region, however the value of K_2 for which the system becomes unstable is well defined by Eq. (25).

A comparison is made for the ITOS value of h, $h = 25$ kg-m/s, for the particular set of gains $K_1 = 5 \times 10^6$ pole-cm/gauss/rad and $K_2 = 5 \times 10^7$ pole-cm-sec/gauss/rad. The results are

$$t_{cn}\text{ (approx.)} = 0.13 \quad t_{c_{01}}\text{ (approx.)} = 2.04 \quad t_{c_{02}}\text{ (approx.)} = 0.68$$
$$t_{cn}\text{ (exact)} = 0.13 \quad t_{c_{01}}\text{ (exact)} = 1.78 \quad t_{c_{02}}\text{ (exact)} = 0.66$$

The agreement is much better in this case because increasing h decreases ε to $\varepsilon = 0.0053$.

Response to Disturbance Torques

The three types of disturbance torques considered are: 1) A constant roll torque which could be the result of solar pressure; 2) A satellite dipole along the pitch axis; and 3) The torque resulting from the effect of the rotation of the Earth on $\dot{B}_\theta$.

1) Let $\bar{\tau}_\phi$ be the constant roll torque. The right-hand side of Eq. (12) becomes $(I_\psi/h^2)\bar{\tau}_\phi$. It is now assumed that this term is small; i.e., $(I_\psi/h^2)\bar{\tau}_\phi = 0(\varepsilon)$. Let

$$\tau_\phi{}^* = \frac{I_\psi}{a\varepsilon h^2}\bar{\tau}_\phi = \frac{\bar{\tau}_\phi}{h\omega_0} \tag{43}$$

then for no secular terms in the solution of ϕ_1 and ψ_1 it is necessary that

$$\begin{aligned} C_{0_\eta} - D_0(1 + K_1{}^* \sin 2\eta) + \tau_\phi{}^* &= 0 \\ D_{0_\eta} + C_0 + K_1{}^*(1+\cos 2\eta)D_0/2 &= 0 \end{aligned} \tag{44}$$

The steady-state solution of this system is an infinite series with the constant term or average given by

$$\begin{aligned} (C_0)_{\text{av}} &= -K_1{}^*\tau_\phi{}^*/2 \\ (D_0)_{\text{av}} &= \tau_\phi{}^* \end{aligned} \tag{45}$$

Thus an approximate solution to a constant roll torque of magnitude $\bar{\tau}_\phi$ is

$$\psi = \frac{-\bar{B}_0{}^2}{2(h\omega_0)^2}(K_1 + 2\omega_0 K_2)\bar{\tau}_\phi \tag{46}$$

$$\phi = \frac{\bar{\tau}_\phi}{h\omega_0}$$

2) Let the satellite pitch dipole be $\bar{M}_\theta$. The resulting disturbance torque $\bar{\tau}$ is

$$\bar{\tau} = \bar{\mathbf{M}}_\theta \times \mathbf{B} = \bar{M}_\theta(-B_\phi\,\mathbf{e}_x + B_\psi\,\mathbf{e}_y) = \bar{\tau}_\psi\,\mathbf{e}_x + \bar{\tau}_\phi\,\mathbf{e}_y \tag{47}$$

Now let

$$\begin{aligned} \bar{\tau}_\psi{}^* = I_\phi\,\bar{\tau}_\psi/h^2, \quad & \tau_\phi{}^* = I_\psi\,\bar{\tau}_\phi/h^2 \\ M_\theta{}^* &= \bar{B}_0\bar{M}_\theta/(h\omega_0) \end{aligned} \tag{48}$$

then the right-hand sides of Eqs. (11) and (12) become

$$\begin{aligned} \tau_\psi{}^* &= -\varepsilon b M_\theta{}^* \cos \bar{\varepsilon t} \\ \tau_\phi{}^* &= -2a\varepsilon M_\theta{}^* \sin \bar{\varepsilon t} \end{aligned} \tag{49}$$

For no secular terms in ψ_1 or ϕ_1 it is necessary that

$$\begin{aligned} C_{0_\eta} - D_0(1 + K_1{}^* \sin 2\eta) - 2M_\theta{}^* \sin \eta &= 0 \\ D_{0_\eta} + C_0 + K_1{}^*(1+\cos 2\eta)D_0/2 + M_\theta{}^* \cos \eta &= 0 \end{aligned} \tag{50}$$

Table 1 Comparison with Floquet theory

$K_1 \times 10^{-7}$	$K_2 \times 10^{-7}$	t_{cn} (approx.)	t_{cn} (exact)	$t_{c_{01}}$ (approx.)	$t_{c_{01}}$ (exact)	$t_{c_{02}}$ (approx.)	$t_{c_{02}}$ (exact)
0.03	0.2	unstable	unstable	5.47	4.99	1.83	1.78
0.03	0.22	66.8	unstable	5.47	4.94	1.82	1.78
0.03	0.4	3.1	3.4	5.41	4.54	1.80	1.72
0.03	0.8	0.99	1.06	5.28	3.85	1.76	1.61
0.03	1.0	0.74	0.79	5.22	3.85	1.74	1.56
0.03	1.8	0.37	0.39	5.00	2.79	1.67	1.39
0.1	0.8	6.1	9.77	1.64	1.46	0.55	0.53
0.1	1.4	0.84	0.92	1.62	1.34	0.54	0.52
0.1	2.0	0.45	0.49	1.60	1.24	0.54	0.50
0.1	3.0	0.26	0.27	1.58	1.1	0.53	0.48
0.1	4.0	0.18	0.19	1.55	0.99	0.52	0.46

Again, the steady-state solution is an infinite trigonometric series and the solution of the coefficients requires the solution of an infinite number of simultaneous equations. An approximate solution is obtained by assuming

$$C_0 = E_1 \cos\eta + E_2 \sin\eta$$
$$D_0 = F_1 \cos\eta + F_2 \sin\eta$$

substituting into Eq. (55) and setting the coefficients of $\sin\eta$ and $\cos\eta$ to zero. The result is

$$C_0 = \frac{4M_\theta^*}{K_1^*}(-K_1^* \cos\eta + \sin\eta)$$
$$D_0 = \frac{4M_\theta^*}{K_1^*}\cos\eta \tag{51}$$

The approximate steady-state solution of ϕ and ψ is

$$\psi = \left(\frac{4\bar{M}_\theta}{\bar{B}_0 \bar{K}_1}\right)\left(\sin\omega_0 t - \frac{\bar{B}_0{}^2}{h\omega_0}\bar{K}_1 \cos\omega_0 t\right)$$
$$\phi = \frac{4\bar{M}_\theta}{\bar{B}_0 \bar{K}_1}\cos\omega_0 t \tag{52}$$
$$\bar{K}_1 = K_1 + 2\omega_0 K_2$$

3) Let the angular velocity of the Earth be ω_e and the additional term in $\dot{B}_\theta$ due to ω_e be $\dot{\bar{B}}_\theta$, then

$$\dot{\bar{B}}_\theta = -B_0 G_2 \omega_e \sin I \sin\gamma \cos u \tag{53}$$

Since u is almost constant over an orbit $\dot{\bar{B}}_\theta$ is almost constant. The strength of the pitch magnet is proportional to $\dot{B}_\theta$, therefore the effect of the rotation of the Earth is the same as that of a constant satellite pitch dipole. Thus Eq. (52) gives the approximate effect of the rotation of the Earth with

$$\bar{M}_\theta = K_2 B_0 G_2 \omega_e \sin I \sin\gamma \cos u \tag{54}$$

Comparison with Numerical Results

A comparison of the approximate solutions for response to disturbance torques with the exact solution obtained by numerical integration of the equations of motion is given in Figs. 2–4. The system parameters used were those of the ITOS satellite. The resulting value of K_1^* is $K_1^* = 0.586$.

The response of the system to a constant roll torque of 20×10^{-8} Newton-meter is given in Fig. 2. The steady-state deviation of the spin axis from the orbit normal is predicted by Eq. (46) is

$$(\psi^2 + \phi^2)^{1/2} = 0.05 \text{ deg} \tag{55}$$

Comparison with Fig. 2 shows that Eq. (46) is a good approximation for the response to a constant roll torque.

The response of the system to a 2500 pole-cm pitch dipole and the effect of the Earth rate on the system is shown in Figs. 3 and 4. The responses predicted by Eq. (52) are

$$(\psi^2 + \phi^2)^{1/2} = 0.74[1 + 0.53 \sin(2\omega_0 t + \alpha)]^{1/2} \text{ deg} \tag{56}$$

and

$$(\psi^2 + \phi^2)^{1/2} = 0.036\,|\cos u|\,[1 + 0.53 \sin(2\omega_0 t + \alpha)]^{1/2} \text{ deg} \tag{57}$$

where α is a phase angle. Comparison with Figs. 3 and 4 shows that Eq. (52) gives an estimate which is about 20% too large for the response to the 2500 pole-cm, pitch dipole and about twice the actual value for the earth rate effect. The larger error in the approximation for the earth rate effect is because the steady-state response has not been attained.

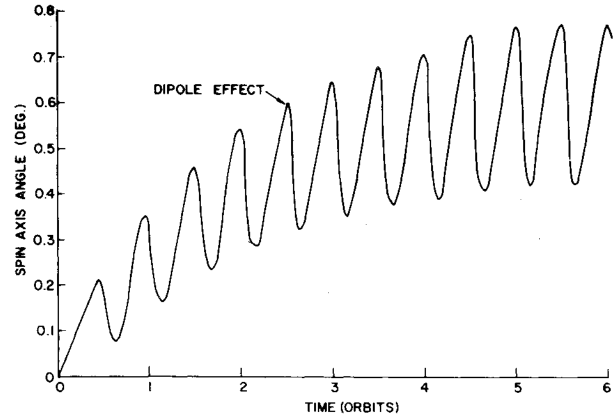

Fig. 3 Effect of a 2500 pole-cm pitch dipole.

Summary

The dynamics of a dual-spin satellite with an attitude control system which utilizes the interaction of the geomagnetic field with a pitch dipole on the satellite has been analyzed by the method of multiple time scales. Approximate time constants and approximate responses to disturbance torques were obtained in terms of the system parameters. Comparison was then made with results from numerical integration of the exact equations of motion.

The effectiveness of this attitude control system is dependent on the strength of the geomagnetic field in the orbital plane. Hence, it is most effective for near magnetic polar orbits and is not recommended for low inclination orbits.

Appendix

Equations of Motion

In this appendix the linearized equations of motion about the nominal state yaw = roll = pitch = O are derived for a rigid body (satellite) moving in a circular orbit about the Earth. The assumptions are: 1) attached to the satellite are a momentum wheel whose axis is coincident with the pitch axis and a magnet aligned with the pitch axis, 2) the only torques acting on the satellite are those resulting from the magnet moving through the magnetic field of the Earth, 3) pitch motion is under control; i.e., $\theta = \dot{\theta} = O$.

Let x, y, z be the principal axes of a rigid body and I_ψ, I_ϕ, I_θ the corresponding principal moments of inertia. The angular momentum of the momentum wheel along the pitch (z) axis is h and $\boldsymbol{\omega}$ is the angular velocity of the body. With $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ as the unit vectors along the x, y, z axes, the angular momentum $\mathbf{H}$ of the system is

$$\mathbf{H} = I_\psi \omega_x \mathbf{e}_x + I_\phi \omega_y \mathbf{e}_y + (I_z \omega_z + h)\mathbf{e}_z \tag{A1}$$

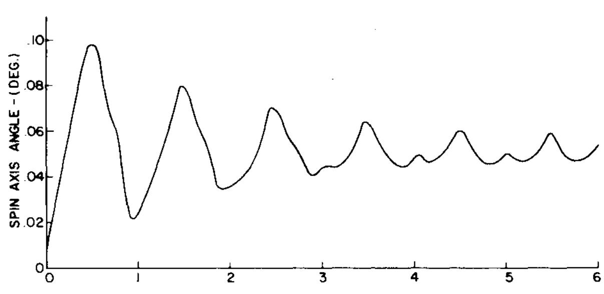

Fig. 2 Response to a constant roll torque of 20×10^{-6} Newton meters.

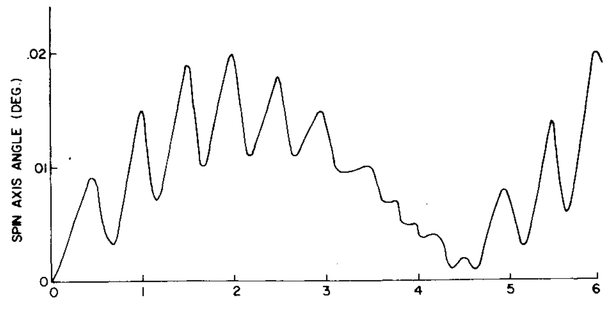

Fig. 4 Effect of earth rotation.

Let

$$\tau = \tau_\psi \mathbf{e}_x + \tau_\phi \mathbf{e}_y + \tau_\theta \mathbf{e}_z \tag{A2}$$

be the torque on the rigid body. The equation of motion of the system is

$$\dot{\mathbf{H}} = \dot{\mathbf{H}}_{REL} + \omega \times \mathbf{H} = \tau \tag{A3}$$

We now need to determine the orientation of the satellite with respect to some known reference frame. Let XYZ be a coordinate system whose origin coincides with the satellite center of mass, the X axis points outward along the radius vector from the center of the Earth to the satellite, the Y axis is in the direction of motion, and the Z axis is normal to the orbit. This reference frame is rotating about the Z axis at the orbital rate ω_o. Let the orientation of the satellite be defined by the standard yaw (ψ), roll (ϕ), and pitch (θ) angles.

For small angles and angular rates, the angular velocity as a function of ψ, ϕ, θ, $\dot{\psi}$, $\dot{\phi}$ and $\dot{\theta}$ is

$$\omega = (\dot{\psi} - \omega_o \phi)\mathbf{e}_x + (\dot{\phi} + \omega_o \psi)\mathbf{e}_y + (\dot{\theta} + \omega_o)\mathbf{e}_z \tag{A4}$$

Substituting Eq. (A4) into (A3), assuming pitch control exists, i.e., $\theta = \dot{\theta} = O$, and linearizing with respect to ψ, ϕ, $\dot{\psi}$ and $\dot{\theta}$ gives

$$I_\psi \ddot{\psi} + (h - I_\phi \omega_0)\omega_0 \psi + [h - (I_\phi + I_\psi)\omega_0]\dot{\phi} = \tau_\psi \tag{A5}$$

$$I_\phi \ddot{\phi} + (h - I_\psi \omega_0)\omega_0 \phi - [h - (I_\phi + I_\psi)\omega_0]\dot{\psi} = \tau_\phi \tag{A6}$$

where h has been replaced by $h + I_\theta \omega_0$.

The torque τ exerted on the satellite by the pitch magnet dipole $\mathbf{M}$ in the Earth's magnetic field $\mathbf{B}$ is

$$\tau = \mathbf{M} \times \mathbf{B} = -M_\theta B_\phi \mathbf{e}_x + M_\theta B_\psi \mathbf{e}_y \tag{A7}$$

where

$$\mathbf{M} = M_\theta \mathbf{e}_z \tag{A8}$$

$$\mathbf{B} = B_\psi \mathbf{e}_x + B_\phi \mathbf{e}_y + B_\theta \mathbf{e}_z \tag{A9}$$

The proposed control law is

$$M_\theta = +K_1 B_\phi \phi - K_2 \dot{B}_\theta \tag{A10}$$

where K_1 and K_2 are the feedback gains, and $\dot{B}_\theta$ is the rate of change of the magnetic field along the pitch (z) axis as sensed by a magnetometer. We now need to consider the magnetic field.

It is assumed that the magnetic field $\mathbf{B}$ can be represented by a dipole which has an inclination of γ with respect to the polar axis. Then

$$\mathbf{B} = \frac{1}{4\pi r^5}[3(\mathbf{r} \cdot \mathbf{M}^*)\mathbf{r} - r^2 \mathbf{M}^*] \tag{A11}$$

where $\mathbf{M}^*$ is the magnetic dipole and $\mathbf{r}$ locates the point at which $\mathbf{B}$ is measured with respect to the dipole. The components of $\mathbf{B}$ along the trajectory axes X, Y, Z are

$$\begin{aligned} B_x &= -2B_0 G_1 \sin(\omega_0 t + \alpha) \\ B_y &= B_0 G_1 \cos(\omega_0 t + \alpha) \\ B_z &= B_0 G_2 \end{aligned} \tag{A12}$$

where it is assumed that at $t = 0$ the satellite is at the ascending node and

$$\begin{aligned} B_0 &= \frac{4\pi}{R^3} \\ G_1{}^2 &= \cos^2 u \sin^2 \gamma + (\sin I \cos \gamma - \cos I \sin u \sin \gamma)^2 \\ G_2 &= \cos \gamma \cos I + \sin \gamma \sin u \sin I \\ \tan \alpha &= \frac{\cos u \sin \gamma}{\sin I \cos \gamma - \cos I \sin u \sin \gamma} \end{aligned} \tag{A13}$$

R is the radius of the orbit, I is the inclination of the orbit, γ is the angle between the magnetic pole and the polar axis, and u is the angle between the projection of the magnetic pole on the equatorial plane and the line of nodes. For a magnetic polar orbit $I = (\gamma + \pi/2)$ and $u = \pi/2$, which gives $G_1 = 1$ and $G_2 = 0$.

Let T_{TB} be the transformation from the trajectory frame (X, Y, Z) to the body frame (x, y, z); i.e.,

$$\mathbf{B}^{\text{BODY}} = T_{TB} \mathbf{B}^{\text{TRAJ}} \tag{A14}$$

For small angles

$$T_{TB} = \begin{pmatrix} 1 & \theta & -\phi \\ -\theta & 1 & \psi \\ \phi & -\psi & 1 \end{pmatrix} \tag{A15}$$

Since we are only considering the small angle (linear) equations B_ψ, and B_ϕ can be replaced by B_x and B_y in Eqs. (A7) and (A10). Now consider $\dot{B}_\theta$.

$$B_\theta = \phi B_x - \psi B_y + B_z \tag{A16}$$

$$\dot{B}_\theta = \dot{\phi} B_x - \dot{\psi} B_y + \phi \dot{B}_x - \psi \dot{B}_y + \dot{B}_z \tag{A17}$$

Neglecting the rotation of the Earth and the regression of the line of nodes due to the nonsphericity of the Earth (these effects will appear as disturbance torques)

$$\begin{aligned} \dot{B}_x &= -2\omega_0 B_0 G_1 \cos(\omega_0 t + \alpha) = -2\omega_0 B_y \\ \dot{B}_y &= -\omega_0 B_0 G_1 \sin(\omega_0 t + \alpha) = \tfrac{1}{2}\omega_0 B_x \\ \dot{B}_z &= 0 \end{aligned} \tag{A18}$$

Substituting Eq. (A18) into Eq. (A17) gives

$$\dot{B}_\theta = (\dot{\phi} - \omega_0 \psi/2) B_\psi - (\dot{\psi} + 2\omega_0 \phi) B_\phi \tag{A19}$$

Substituting Eq. (A19) into Eq. (A10) and the result into Eq. (A7) gives the following equations of motion

$$\begin{aligned} &I_\psi \ddot{\psi} + K_2 B_\phi{}^2 \dot{\psi} + (h - I_\phi \omega_0 + \tfrac{1}{2} K_2 B_\phi B_\psi)\omega_0 \psi + \\ &\quad [h - (I_\phi + I_\psi)\omega_0 - K_2 B_\phi B_\psi]\dot{\phi} + \\ &\qquad (K_1 + 2\omega_0 K_2) B_\phi{}^2 \phi = 0 \end{aligned} \tag{A20}$$

$$\begin{aligned} &I_\phi \ddot{\phi} + K_2 B_\psi{}^2 \dot{\phi} + [h\omega_0 - I_\psi \omega_0{}^2 - (K_1 + 2\omega_0 K_2) B_\phi B_\psi]\phi - \\ &\quad [h - (I_\phi + I_\psi)\omega_0 + K_2 B_\phi B_\psi]\dot{\psi} - \tfrac{1}{2} K_2 \omega_0 B_\psi{}^2 \psi = 0 \end{aligned} \tag{A21}$$

Equations (A20) and (A21) are 2 second-order linear differential equations with periodic coefficients with a period of half the orbital period.

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